A New Approach to the Design of Graded-Index Guided Wave Devices

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Abstract—**An inverse scattering approach to modeling single mode gradient-index planar guided wave devices is presented. The method involves solving the Gelfand–Levitan–Marchenko integral equation to obtain a refractive index profile which is infinite in extent. The theory is developed to account for truncations of this refractive index profile, illustrating the effects of a finite core width upon the propagation constant. Refractive index profiles with symmetric and asymmetric cladding indexes are discussed.**

THE DESIGN of GRIN devices involves synthesis of refractive index profiles. In the standard direct methods, a refractive index profile is specified *a priori* and the quantities of interest, usually the propagation constants and the mode functions, are calculated using either analytical or numerical methods. A limited number of refractive index profiles, such as the parabolic, exponential, and hyberbolic secant types, allow for exact analytic solutions when they are considered infinite in extent, but it is necessary to resort to numerical or perturbation techniques to analyze the effects of a finite core width, which is a necessary part of any practical design. We discuss here a new approach, namely an inverse scattering method, in which the refractive index profiles are reconstructed from the required device characteristics, such as propagation constants and mode functions. The method allows for efficient, exact analysis of the effects of truncating the refractive index profile.

The propagation of TE modes in a planar GRIN device is governed by a Schroedinger-type equation [1],

$$\frac{d^2\Psi}{dx^2} + \left[k^2 - v(x)\right]\Psi = 0. \tag{1}$$

Here, $\Psi \equiv E_y(x)$, $k^2 = k_0^2 n_2^2 - \beta^2$ and $v(x) \equiv k_0^2[n_2^2 - n^2(x)]$, where n_2 represents the refractive index of the cladding, $n(x)$ the core index profile, and β the longitudinal propagation constant(s). Propagation in the z direction has been assumed. Note that $v(x)$ vanishes in the cladding; at this stage it is assumed the core is bounded by claddings of equal refractive index.

According to inverse scattering theory [2], the potential $v(x)$ can be reconstructed from

$$v(x) = 2\frac{d}{dx}K_-(x, x), \tag{2}$$

where the kernel $K_-(x, t)$ satisfies the Gelfand–Levitan–Marchenko (GLM) integral equation:

$$K_-(x, t) + R_-(x + t) + \int_{-\infty}^{x} K_-(x, \zeta)R_-(\zeta + t)\, d\zeta = 0. \tag{3}$$

The scattering data completely determines the reflected transient or characteristic function $R_-(x + t)$:

$$R_-(x, t) = \sum_{n=1}^{N} d_n^2 \exp\left[\kappa_n(x + t)\right] + \frac{1}{2\pi}\int_{-\infty}^{+\infty} r_-(k)\exp\left[-ik(x + t)\right]dk, \tag{4}$$

where $r_-(k)$ is the reflection coefficient for a plane wave incident on $v(x)$ from $x = +\infty$, and the set of d_n's are the normalization constants for the N bound states (i.e., propagating modes) whose eigenvalues are represented by the discrete set $k_n = i\kappa_n$ ($\kappa_n > 0$), for $n = 1, 2, \cdots, N$. (For a wave incident from $x = -\infty$, the potential is characterized by a corresponding reflection coefficient $r_+(k)$; knowledge of either reflection coefficient is sufficient to reconstruct the potential.) The corresponding transmission coefficient $t_-(k)$ $(= t_+(k) = t(k))$, whose poles on the positive imaginary axis are the set of eigenvalues $\{k_n\}$, provides the propagation constants for the guided modes. For arbitrary values of k, the Schroedinger equation admits two Jost solutions, denoted $f_\pm(k, x)$, each with the asymptotic behavior

$$\lim f_\pm(k, x) = e^{\pm ikx}, \quad x \to \pm\infty. \tag{5}$$

The functions $f_\pm(k_n, x)$ are proportional to the bound state wavefunctions of the Schroedinger equation.

As an example of the GLM reconstruction method, consider the single-mode ($N = 1$) reflectionless potential with scattering data $r(k) = 0$, $d_1 = \sqrt{2}$, $\kappa_1 = 1$. The reconstructed potential, resulting from the straightforward application of (2)–(4), is $v(x) = -2\ \mathrm{sech}^2 x$. In the context of optical waveguides, this potential has been extensively studied using direct methods [3]. The Jost solutions corresponding to this potential take the form [4]

$$f_\pm(k, x) = e^{\pm ikx}\left[\frac{ik \mp \tanh x}{ik - 1}\right] \tag{6}$$

Note that $\mathrm{sech}^2(x)$ approaches zero only as $x \to \pm\infty$, necessitating, in practical devices, truncation of the refractive index profile at $x = x_1$, x_2 as shown in Fig. 1. (The values x_1, x_2 may be positive or negative, the only restriction being $x_1 \le x_2$). These truncations represent core-cladding interfaces, which will

Manuscript received December 11, 1990.

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IEEE Log Number 9143087.

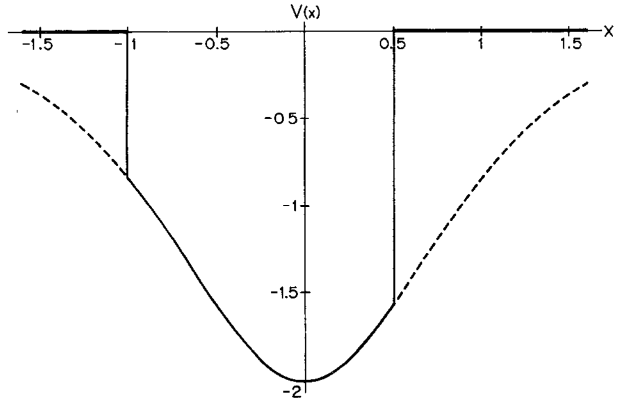

Fig. 1. Untruncated reflectionless potential $v(x) = -2 \operatorname{sech}^2 x$ (dashed line). Truncated potential (solid line) shown for $x_1 = -1$, $x_2 = 0.5$.

alter the scattering data, reducing the magnitude of k_1, since the mode is less tightly bound, and altering $r_\pm(k)$ since the potential is no longer reflectionless.

The transmission coefficient for a structure truncated at points x_1 and x_2 may be written in terms of the Jost solutions for the original untruncated structure. While potentials truncated at a single point have been considered in the applied mathematics literature [5], the method is extended here to take into account a double truncation, which is representative of a guided-wave structure. For these purposes, it is useful to consider the integral representation of the Jost solutions [6]

$$f_\pm(k, x) = e^{\pm ikx} \mp \int \frac{\sin k(x - z')}{k} v(z') f_\pm(k, z')\, dz', \tag{7}$$

where the limits on the integral are $[-\infty, x]$ and $[x, \infty]$ for $f_-(k, x)$ and $f_+(k, x)$, respectively. Relations between the Jost solutions $f_+(k, x)$ and $f_-(k, x)$,

$$f_\pm(k, x) = \frac{1}{t(k)} f_\mp(-k, x) + \frac{r_\mp(k)}{t(k)} f_\mp(k, x), \tag{8}$$

lead to the Wronskian relations

$$\frac{2ik}{t(k)} = W[f_-(k, x), f_+(k, x)] \tag{9}$$

and

$$2ik \frac{r_\pm(k)}{t(k)} = \mp W[f_\mp(k, x), f_\pm(-k, x)] \tag{10}$$

where $W[f, g] \equiv fg' - gf'$, the prime denoting differentiation with respect to x.

Equations (7)–(10) provide all the relationships needed to consider the effects of core-cladding interfaces. First, consider a truncation at a single point x_1. Equation (9) provides an expression for the transmission coefficient of this singly truncated potential,

$$t^T(k) = 2ik\, e^{ikx_1}[f'_+(k, x_1) + ikf_+(k, x_1)]^{-1}, \tag{11}$$

where the superscript T is used to denote a single truncation. Equation (11) derives from the fact that $f_-^T(k, x) = e^{-ikx}$ for $x \le x_1$, whereas for $x \ge x_1$, $f_+^T(k, x) \equiv f_+(k, x)$, both of which are evident from (7). Observe that the Wronskian is independent of the coordinate, allowing evaluation at any value, in this case x_1. Using similar reasoning, note that an additional truncation, this time at x_2, produces a doubly truncated potential with a transmission coefficient of the form

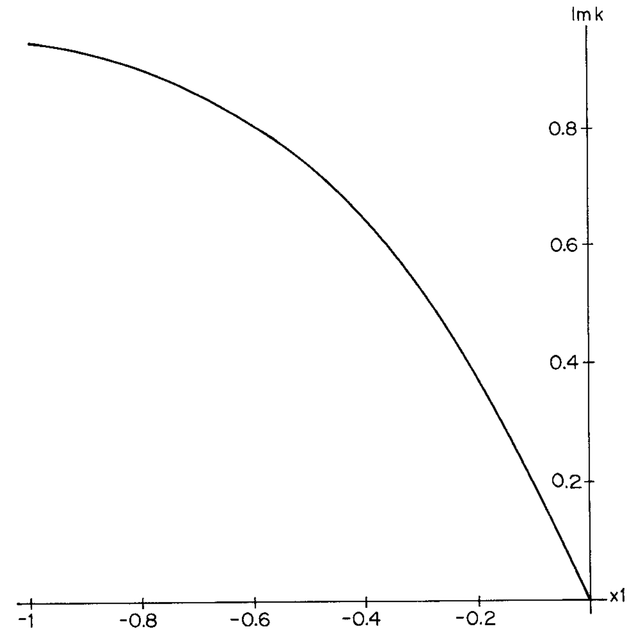

Fig. 2. Variation of pole position representing the propagating mode, as core width is narrowed, (for case $x_2 = -x_1$).

$$t^{TT}(k) = -4k^2 e^{-ik(x_2 - x_1)}[f_a f_b - f_c f_d]^{-1}, \tag{12}$$

derived with the help of (8)–(10), where

$$f_a \equiv f'_+(k, x_1) + ikf_+(k, x_1),$$
$$f_b \equiv -f'_+(-k, x_2) + ikf_+(-k, x_2),$$
$$f_c \equiv f'_+(-k, x_1) + ikf_+(-k, x_1)$$

and

$$f_d \equiv -f'_+(k, x_2) + ikf_+(k, x_2).$$

Observe that $t^{TT}(k)$ is now written in terms of the Jost solution $f_+(k, x)$ of the simpler untruncated structure.

The width of the waveguide is controlled by the parameters x_1 and x_2. As these are varied, causing a change in the core width, the poles of $t^{TT}(k)$ move in the complex k plane. As an example, consider truncation of $v(x) = -2 \operatorname{sech}^2 x$. Calculation of $t^{TT}(k)$, using (6), yields two poles—one each on the upper and lower *Im k* axes—the former representing the guided mode. Fig. 2 illustrates the behavior of this bound state for the case of a symmetric truncation at points x_1 and x_2 such that $x_1 = -x_2$. Only as $x_1 \to 0$ does k_1 vanish, indicating that a propagating mode exists for any finite core width. The act of truncation introduces a pole on the negative portion of the *Im k* axis, whose behavior is illustrated in Fig. 3. As a test of this method, it can be shown that as $x_1 \to \infty$, $t^{TT}(k) \to (k + i)/(k - i)$, the transmission coefficient of the untruncated structure [7].

In instances where $v(x)$ approaches different values as $x \to \pm\infty$, $t_+(k) \ne t_-(k)$. However, both exhibit the same bound state poles, so either one is suitable for our purposes. This situation is encountered in examples of GRIN devices where the refractive indices of the regions bounding the core are different. Suppose that instead of vanishing in the region $x \ge x_2$, one has $v(x) = \bar{v}$, where $\bar{v} \ge 0 \ge v(x_2)$. Defining the quantity $\bar{k}$

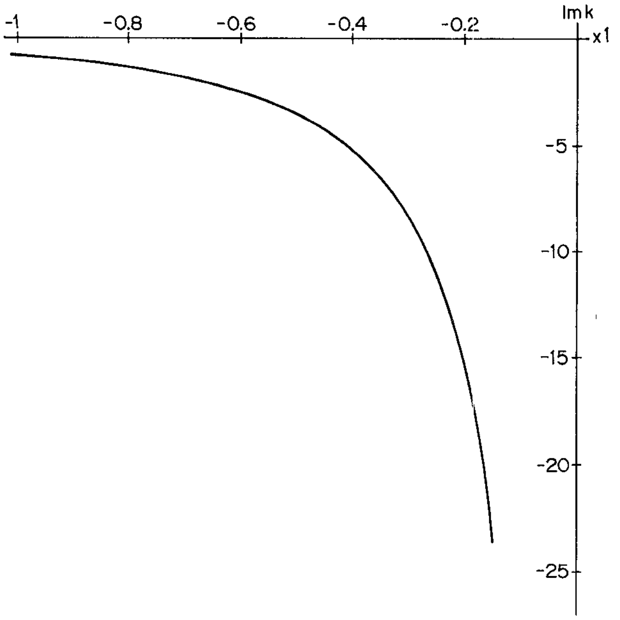

Fig. 3. Variation of position of additional pole in lower half plane for same case as in Fig. 2.

$\equiv \sqrt{(k^2 - \bar{v})}$, one can proceed to calculate $t^{TT}(k)$ by suitably modifying (12):

$$t_-^{TT}(k) = -4k^2 e^{-i(\bar{k}x_2 - kx_1)}\left[f_a \bar{f}_b - f_c \bar{f}_d\right]^{-1}, \quad (13)$$

where

$$\bar{f}_b \equiv -f'_+(-k, x_2) + i\bar{k} f_+(-k, x_2)$$

and

$$\bar{f}_d \equiv -f'_+(k, x_2) + i\bar{k} f_+(k, x_2).$$

Fig. 4 illustrates pole behavior for an example in which the guiding region is bounded by core-cladding interfaces at $x_2 =$ 0.5 (fixed) and at x_1 (variable). Assuming that $\bar{v} = +1.4$ (and $v(x) = 0$ for $x < x_1$), calculations yield a pole that moves down the axis as $x_1 \rightarrow 0.5$. At $x_1 \simeq -0.02$, the pole moves onto the negative portion of the axis, indicating the core width at which the guide no longer supports a propagating mode, confirming the results of direct methods that have shown that in the case of asymmetric cladding indices, the propagating mode disappears at a nonzero core width [8].

In conclusion, this method provides a viable procedure for calculating the exact value of the propagation constant for a finite width refractive index profile. The technique presented here is completely general, allowing extension to multimode waveguides and various other refractive index profiles.

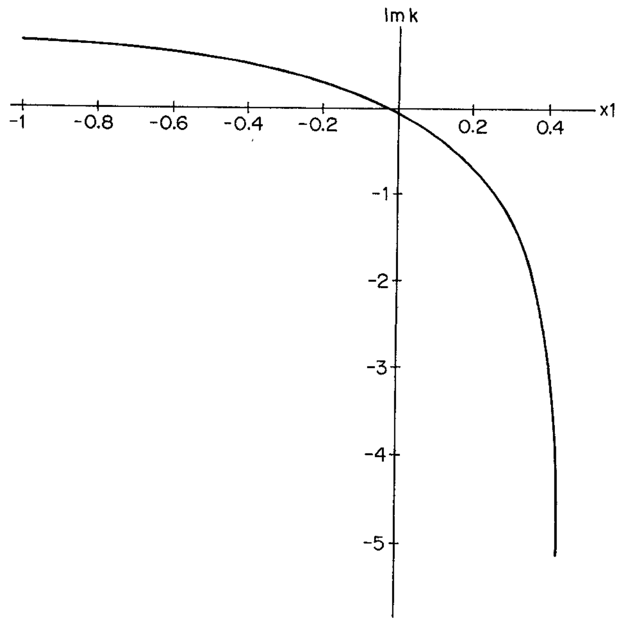

Fig. 4. Transformation of propagating mode from bound state (upper k axis) to unbound condition (lower k axis) with narrowing of core. (Asymmetric cladding indexes, x_2 fixed at 0.5).

Acknowledgment

L. S. Tamil would like to thank Dr. A. K. Jordan with the Naval Research Laboratory, for many helpful discussions.

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